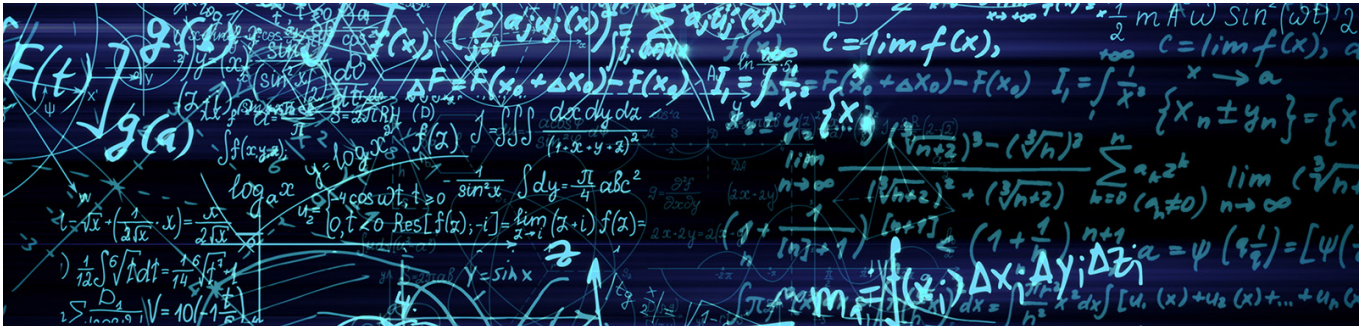


# Overview

## The Sauerbrey Equation

A brief explanation of what it is and how it is derived



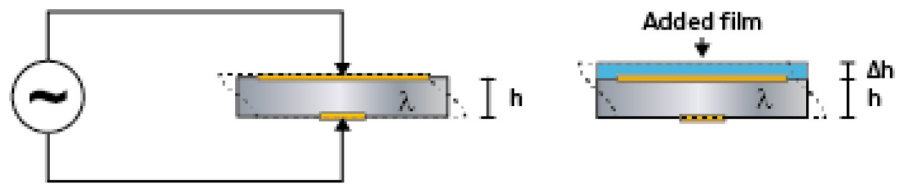
QCM-technology is essentially a balance for very small masses. The ability to measure mass changes originates in the discovery of the relation between the QCM crystal resonance frequency and its mass. This relation, which is known as the Sauerbrey equation, was formulated in 1959 by Günter Sauerbrey, at the time a Ph.D. student in Berlin.

### Converting frequency change to mass change

The Sauerbrey equation is a linear relationship between the resonance frequency changes of an oscillating quartz crystal and its mass changes, Eq. 1.

$$\Delta m = -C \cdot \frac{\Delta f}{n} \quad (1)$$

In practice, this equation can be used to calculate the mass of thin layers added to a QCM crystal surface. By inserting the resonance frequency change,  $\Delta f$ , that results from the layer addition, into Eq. 1, you will get the mass,  $\Delta m$ , of the added layer. The constant,  $C$ , is called the sensitivity constant. It is related to the properties of quartz.



**Figure 1:** (Left) Schematic illustration of a quartz crystal, of thickness  $h$  and mass  $m$ , excited to resonance by the application of an alternating voltage. (Right) An oscillating crystal with added mass. Now the thickness is  $h + \Delta h$  and the mass is  $m + \Delta m$ .

For a 5 MHz crystal,  $C$  equals 17.7 ng/(cm<sup>2</sup>·Hz). The parameter  $n$  is the number of the odd harmonic, and can be 1, 3, 5, 7, ... etc.

### The relation explained

The linear frequency-mass relation is based on the behavior of a pure quartz crystal, where a crystal of thickness  $h$  (and mass  $m$ ), illustrated in Fig 1, will have a certain resonance frequency,  $f_0$ . Now, we picture a scenario where a layer of material, different from quartz, is added to the crystal surface, Fig 1, right. If certain conditions are fulfilled, this scenario can be approximated with that of a "thicker crystal"-scenario, i.e. a crystal with thickness  $h + \Delta h$  (and mass  $m + \Delta m$ ). Such a crystal will have a resonance frequency  $f < f_0$ .

The model assumes that the layer added to the crystal surface can be approximated

to be a part of the oscillating crystal itself. It is therefore important that the situation is similar enough to the "pure-quartz" scenario. For this to hold, the layer on the sensor must be thin, rigid and firmly attached to the crystal surface. If these conditions are fulfilled, the Sauerbrey equation could be used to calculate the mass of the added layer. If, however, the layer on the sensor is soft, thick or not coupled to the sensor surface, the Sauerbrey equation is not valid. In these situations, the model will fail, and the calculated mass will be underestimated.

### Deriving the equation

Resonance in an AT-cut piezoelectric quartz crystal will occur when the thickness of the crystal is an odd integer of half wavelengths,  $\lambda$ , of the induced wave, and

the oscillation will have its anti-nodes at each surface, Fig 1. The resonant frequency can then be stated as:

$$f = n \frac{v_q}{2t_q} = n \cdot f_0 \quad (2)$$

where the included parameter are described below. Below are also listed additional parameters that are used in the following derivation of the equation.

$f$  – resonance frequency, 1/s

$f_0$  – fundamental resonance frequency, 1/s

$n$  – harmonic number

$v_q$  – wave velocity in quartz plate<sup>1</sup>, m/s

$t_q$  – thickness of quartz plate, m

$\rho_q$  – density of quartz<sup>2</sup>, kg/m<sup>3</sup>

$M$  – total added mass, kg

$A$  – Active area, m<sup>2</sup>

$m_q$  – Areal mass, kg/m<sup>2</sup>

The fundamental resonant frequency,  $f_0$ , where  $n = 1$ , can be expressed as:

$$f_0 = n \frac{v_q}{2t_q} \quad (3)$$

The mass per area can be expressed as the product of thickness and density, if evenly distributed:

$$\frac{M}{A} = m_q = t_q \cdot \rho_q \quad [m \cdot \frac{kg}{m^3} = \frac{kg}{m^2}] \quad (4)$$

by inserting (4) into equation (3) and differentiating the expression, it becomes:

$$df = -\frac{f}{m_q} dm_q \quad (5)$$

The change in added mass,  $m$ , on the crystal can be treated as an equivalent of the crystal itself, provided that the added mass is:

- (i) small compared to the crystal mass
- (ii) rigidly adsorbed, with no slip or deformation imposed by the oscillating surface
- (iii) evenly distributed over the surface

By letting  $d \rightarrow \Delta$ , and using equations 2, 3 and 4 to replace  $m_q$  and  $f$  in equation (5) we get:

$$\begin{aligned} df = \Delta f &= -\frac{f}{t_q \cdot \rho_q} \Delta m = \\ &= -n \cdot \frac{2f_0^2}{v_q \cdot \rho_q} \Delta m = -n \cdot \frac{1}{C} \Delta m \end{aligned} \quad (6)$$

The constant  $C$  is the so-called mass sensitivity constant. By performing a unit analysis of equation (6), the unit of  $C$  equals:

$$C = \frac{v_q \cdot \rho_q}{2f_0^2} \quad [\frac{m \cdot kg}{s \cdot m^3} = \frac{kg \cdot s}{m^2}] \quad (7)$$

When equation (6) is used to calculate mass changes on a crystal, it can be rewritten on the following form:

$$\Delta m = -C \cdot \frac{\Delta f}{n} \quad (8)$$

in which case  $m$  is the areal mass difference, and eq. (8) is the so-called Sauerbrey equation.

### Active area

By using eq. (8) for estimations of added mass, it is not necessary to know the active area as long as the assumptions (i, ii, iii) are fulfilled.

### References

Original work by Sauerbrey: Sauerbrey, Z. Phys. **1959**, 155, 206-222.

For a modern explanation (in English): Höök, F. Development of a novel QCM technique for protein adsorption studies, **1997**, Chalmers University of Technology, Department of Biochemistry & Biophysics and Department of Applied Physics, PhD Thesis.

Footnote:

<sup>1</sup>Wave velocity in an AT-cut quartz plate,  $v_q = 3340$ m/s

<sup>2</sup>Density of AT-cut quartz  $\rho_q = 2650$  kg/m<sup>3</sup>

